Mechanical Efficiency of Kinematic Heat Engines

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ABSTRACT: *A general theory of the mechanical efficiency of kinematic heat engines is presented. Concepts are introduced that precisely relate mechanical efficiency to general characteristics of an engine. Theorems are proven that lead to a mathematically explicit limit to the mechanical efficiency of reciprocating engines. This limit is the mechanical analogue of the Carnot limit to thermal efficiency. The ideal Stirling engine plays a key role in this theory.*

I. Introduction

This paper presents a general conceptual and basic quantitative analysis of the mechanical efficiency of kinematic heat engines. Typically, engineering studies of the mechanical efficiency of heat engines are made on a case by case basis. In ordinary practice, kinematic analysis and computer simulation of specific engine mechanisms coupled with calculated or measured pressure–volume cycles usually can be effectively used for evaluating and optimizing engine designs. Nevertheless, an overall insightful view of mechanical efficiency would be of great value in practical settings and it is of obvious importance for theoretical work. However, no general treatment of mechanical efficiency of heat engines has been available.

This is in sharp contrast to the situation regarding the thermal efficiency of heat engines. Classical thermodynamics treats the subject of thermal efficiency in great generality. Its results, although obtained in a highly idealized setting, are of profound importance to engine theorists, designers, and operators. This paper approaches mechanical efficiency of engines at a similar level of ideality and generality.

The paper first identifies and introduces a conceptual basis for the essential energy transfers which take place among the basic components of reciprocating heat engines. This illuminates the interplay of engine characteristics quantitatively determining mechanical efficiency. Second, it is shown that the ideal Stirling engine has the highest mechanical efficiency potential; this yields an absolute upper limit to the mechanical efficiency of all kinematic engines. This limit is given explicit mathematical expression in terms of only three basic engine parameters: the ratio of the temperature extremes between which the engine operates, the volume compression ratio and the effectiveness of the mechanism of the engine in transporting energy between the piston and output shaft. This result is the mechanical correlate of the Carnot limit to thermal efficiency.

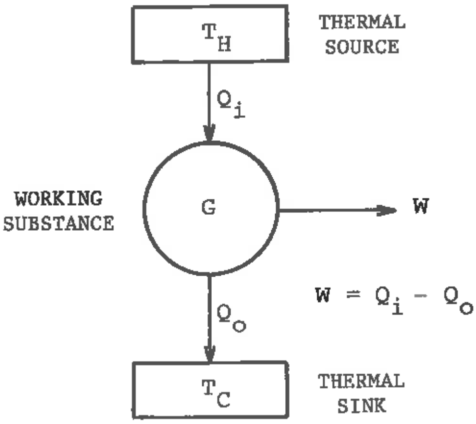

FIG. 1. Elemental heat engine diagram depicting heat transfers Q_i and Q_o to and from the working substance and cyclic work output W.

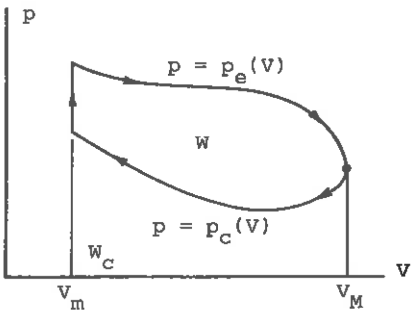

FIG. 2. A regular cycle in the p–V plane.

II. Basic Engine Concepts

Figure 1 is a typical textbook schematic representation of a heat engine. The body of working substance is represented by G, and T_H and T_C are the temperatures of the heat source and sink respectively. The net work done by the working substance per cycle, W, is the difference $Q_i - Q_o$ between the heat absorbed per cycle from the high temperature source and the heat rejected to the lower temperature reservoir.

Figure 2 depicts in the p–V plane a cycle of the working substance of a heat engine. The characteristics of this cycle are typical of all of the well known theoretical cycles (Carnot, Otto, Stirling, etc.) as well as of cycles encountered in practice. Such cycles will be referred to as regular cycles.

A. *Regular cycles*

By a *regular cycle* in the p–V plane is meant a pair of functions p_c and p_e defined and continuous on a closed bounded interval $I = [V_m, V_M]$ and satisfying $p_e(V) \geqslant p_c(V)$ for all V in I. The function p_e will be referred to as the *expansion* *process pressure* and p_c as the *compression process pressure*. The definition formalizes well-behaved cycles which are simple closed curves in the p–V plane and which have exactly two volume reversals, one at each volume extreme. To visualize the engine cycle represented, p_e is traced out from left to right (i.e. in the direction of increasing volume) and p_c in the opposite direction (decreasing volume). If at either endpoint of the cycle the pressure functions do not coincide, a vertical segment corresponding to a constant volume process in the operation of the engine is understood to occur. The vertical segments are directed to agree with the sense of the curves connected so that a simple closed clockwise oriented curve results representing an engine cycle. Heat pump cycles would have the opposite orientation, but the discussion here will be limited to engine cycles; it will be obvious how the results obtained here can be applied to heat pumps if desired.

It might be well to point out here that all the results below apply with virtually no modification to cycles in which p_e and p_c are only piecewise continuous. This more general concept models cycles in which constant volume processes can occur at locations other than at the volume extremes. In such cases vertical segments are understood at the jump discontinuities of the pressure functions as well as at the volume extreme gaps. However, such cycles are only occasionally encountered or needed, so that it seems unnecessary to burden the present development with the added complication. The same applies all the more to cycles which are non-simple (i.e. self-intersecting) or which have intermediate volume reversals.

Although the diagram of Fig. 1 is entirely adequate for the purpose of discussing the *thermal efficiency* $\eta_t = W/Q_i$ of the cycle, it does not depict the fact that work must be done on the engine fluid to carry out the cycle. This is quite visible in Fig. 2 and represented by the area labelled W_c below the cycle; this will be called the *compression work* of the cycle:

$$W_c = \int_{V_m}^{V_M} p_c(V)\,\mathrm{d}V.$$

The *expansion work* done by the engine in each cycle is $W_e = W + W_c$. Equivalently,

$$W_e = \int_{V_m}^{V_M} p_e(V)\,\mathrm{d}V.$$

Figure 3 shows these essential work transfers. To realize a self-acting engine, means must be provided to divert and store some of the cyclic expansion work output W_e, namely the compression work W_c, and to redirect it to the engine substance during appropriate parts of the cycle. This can be accomplished in any number of ways in practice, depending upon the type of engine. The type of engine with which this paper deals is based on the common piston–crankshaft variety, conceptually described below and hereafter referred to as a kinematic heat engine.

B. *Kinematic heat engines*

A *kinematic heat engine* is characterized by the elements schematically represented in Fig. 4. The working substance, typically a gas, is contained in a capsule called the workspace which is equipped with a means for varying the volume,

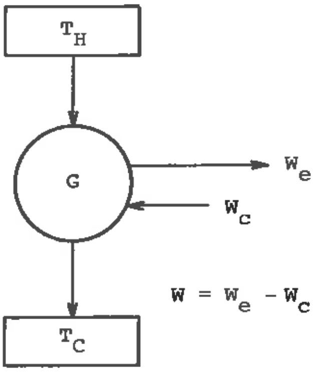

FIG. 3. Heat engine diagram showing cyclic work transfers W_c and W_e to and from the working substance.

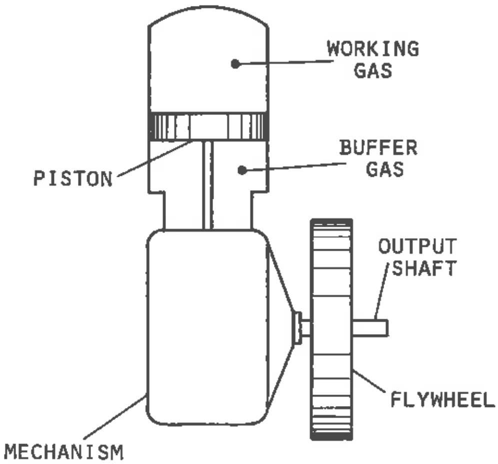

FIG. 4. The elements of a kinematic engine.

usually a piston. In the engines analyzed in this paper, only a single body of *working gas* will be assumed. This does not represent a significant loss of applicability since many multi-cylinder engines can be considered to be parallel connections of single workspace engines. However, more than one piston may be used to vary the single workspace volume as in certain types of Stirling engines (1). The word *piston* will be used in a generic way to signify equivalent positive displacement devices such as bellows or diaphragms. It will also be assumed for convenience that the working gas is ideal.

The workspace is also equipped with means to thermally interact with the necessary heat reservoirs. These are not shown in the figure because they do not directly influence the mechanical efficiency of kinematic engines. They of course shape the thermal processes that take place in the workspace, and this determines the p–V diagram which is here taken as the starting point for the analysis of mechanical efficiency. The only assumption made about the thermodynamic processes occurring in the engine is that the resulting cycle is regular in the sense of the definition above.

The prime characteristic of the kinematic engine is the *mechanism* linking the piston to the output shaft. The link is a kinematic one so that the motion of the piston, and all other moving parts of the engine, is completely constrained by the mechanism. The mechanism transmits force as well as motion and so is a "machine" in the parlance of the kinematician, but the term mechanism will be used here to avoid confusion with the engine as a whole. Note that since the motion of the piston is fixed by the mechanism, so are the volume extremes of the engine's cycle, that is, the volume extremes are characteristics of a given engine.

The workspace usually contains other kinematic devices not shown in Fig. 4 in the interest of simplicity and universality. These items include any valves, displacers, etc. which are necessary to carry the working fluid through the desired thermodynamic cycle. These devices, as well as auxiliary pumps, fans, etc. are kinematically linked to and are driven by the mechanism and are conceptually considered as part of the mechanism for the purposes of this study.

In typical turbine type heat engines, the expansion and compression processes (indeed, all processes) take place simultaneously in different locations in the engine, and the processes are continuous. In the kinematic engines being modelled here, the processes are discrete and sequential. Because of this, a kinematic engine must be equipped with a work reservoir. For single-workspace engines with a rotating shaft output, this reservoir invariably takes the form of a *flywheel* as Fig. 4 depicts. Other devices can be used, for example, pendulums or springs on engines with oscillating or reciprocating output. In multi-workspace engines, each workspace can use some of the others for this purpose. Under steady state operation, the flywheel does not experience a net gain in energy over a cycle. During each cycle, it absorbs, stores, and returns the energy to the engine that is necessary to complete the cycle; the remainder is directed through the output shaft for use outside the engine. For simplicity, one flywheel located on the output shaft as Fig. 4 indicates will be assumed.

C. *Buffer pressure*

The single-workspace kinematic engine needs nothing more in principle than the features described above, but in practice it usually has a near constant external pressure acting on the non-workspace side of the piston. The source of this presssure is usually due to the surrounding atmosphere but sometimes a special enclosure called a *buffer space* is provided to permit the use of elevated pressure (**2**). As will be seen, the *buffer pressure*, whatever its source, plays an extremely important role in the overall mechanical efficiency of a kinematic engine.

All work transferred through the engine mechanism is subject to some loss due to friction. This applies to transfer in both directions: from flywheel to piston as well as from piston to flywheel and output shaft. The works transferred are not generally W_e and W_c because of the action of the buffer pressure. The buffer gas, like the flywheel, absorbs, stores, and returns energy to the working gas during the cycle. But since it acts directly on the piston, it diverts and recycles some work

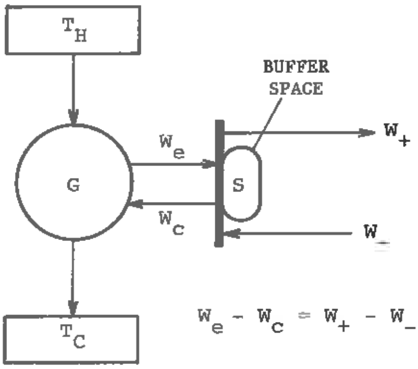

FIG. 5. Engine diagram showing the influence of buffer pressure on external work transfers.

away from the mechanism. Figure 5 represents the buffer space by the element labeled S. The work quantities that must be mechanically transmitted to and from the engine piston are reduced from W_e and W_c on the workspace side of the piston to W_+ and W_- by the influence of the buffer gas pressure. This reduces the friction losses in the mechanism section of the engine in a way to be precisely described below. We refer to W_+ and W_- as the positive and negative *piston work* respectively; W_+ is the non-negative work done on the mechanism by the piston, and W_- is the non-negative work done to the piston by the mechanism in each cycle. Note that the arrows in Fig. 5 refer to positive work transfer only and not necessarily to piston motion or force direction. These concepts are explained in more detail below.

Since the buffer pressure acts directly on the piston with no intervening mechanism, it does not suffer the kind of frictional loss that flywheel energy storage does. There is piston seal friction, but this is most appropriately included in the friction of the mechanism section and not associated with the buffer space. The buffer gas may however suffer a loss in another way when the buffer space volume is finite. This loss is termed *hysteresis* or *transient heat transfer* loss (3). It occurs when the pressure of the buffer space fluctuates with the volume changes induced by piston motion. In the interior of the buffer space, the gas experiences a corresponding temperature fluctuation. Near the walls of the buffer space, conditions are nearly isothermal. This produces a net flow of energy from the buffer gas to the container walls over each cycle. To counter this in practice, one makes the volume of the buffer space as large as practical. This minimizes the pressure excursion and therefore minimizes the hysteresis loss. Usually the pressure fluctuation in the buffer space can be made relatively small. Accordingly, for the remainder of this paper, we shall idealize and assume that buffer pressure is constant, just as it is when the atmosphere serves as the buffer gas. Therefore, the *buffer gas* acts as a constant pressure lossless energy reservoir in direct communication with the piston. In other words, the difference between the positive and the negative piston work is exactly the indicated work of the cycle:

$$W_+ - W_- = W = W_e - W_c.$$

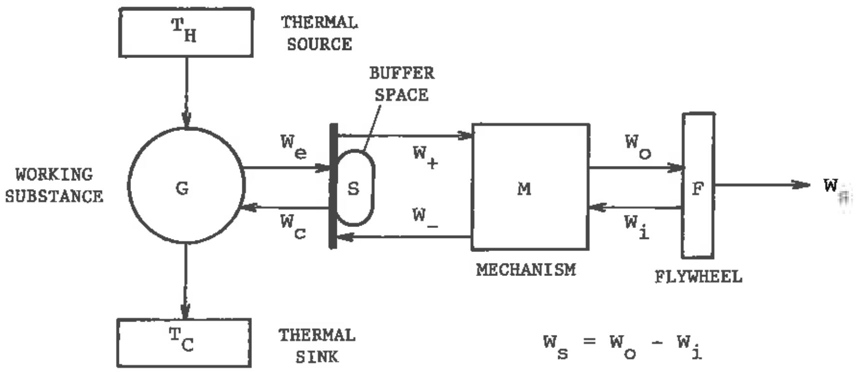

FIG. 6. Diagram of a kinematic heat engine showing the work transfers between the components.

D. *Shaft work*

The complete kinematic engine as just described can be conceptually represented as in Fig. 6. Device M represents the mechanism and F the flywheel; W_s is what is usually called the *cyclic shaft work*. In practical terms, W_s is the "useful" work "coming out" of the engine in each cycle. It is the difference between the cyclic work W_o received by the flywheel/output shaft through the mechanism, and the cyclic work W_i taken from the flywheel and directed into the mechanism to sustain operation.

W_o is the positive piston work W_+ reduced by the friction losses in the mechanism. W_i is the work that must be fed into the mechanism from the flywheel in order to produce the negative piston work W_- when reduced by the friction losses incurred in transmission through the mechanism to the piston. Therefore W_o and W_i depend upon the effectiveness of the mechanism of the engine considered as a simple machine.

III. Mechanical Efficiency Concepts

A. *Mechanism effectiveness*

For any given engine mechanism at each position and for any given input force applied there, the ratio of the corresponding output force to the ideal output force (i.e. the output force if all friction were absent) we will call the instantaneous *mechanism effectiveness* (**4**). This definition is essentially nothing more than the instantaneous version of the usual elementary textbook definition of the efficiency of a machine. We justify using the term effectiveness here on the grounds that it helps to emphasize and isolate the role that the basic performance of the engine's machinery plays in the overall mechanical efficiency of the whole engine. The effectiveness of a mechanism is always of course at most unity.

The effectiveness is generally a function of velocity as well as position and load. Velocity consideration is especially important when high speeds or relatively large moving masses are involved. However, such considerations are highly design and

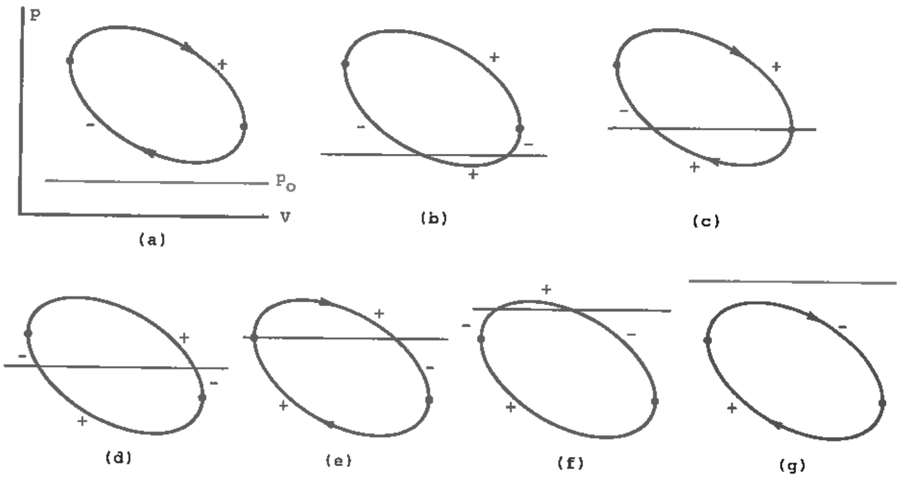

FIG. 7. An engine cycle shown with various buffer pressure levels. The signs indicate the direction of energy transfer between the piston and the flywheel.

material specific and therefore are not amenable to general treatment. The aim here is for very general results about mechanical efficiency so we shall adopt the strategy of ignoring velocity effects in the derivation of our results and keep in mind the fact that they apply only to cases where velocity is the same, or constant, or where it is relatively unimportant as, for example, when speeds are low. More detailed analyses of course must include velocity effects not only on the mechanism effectiveness but also on the thermodynamic cycle (5).

B. *Definition of mechanical efficiency*

The *mechanical efficiency* of a kinematic heat engine is defined as the ratio of the cyclic shaft work to the indicated cyclic work:

$$\eta_m = W_s/W.$$

In this general model the buffer space is ideally lossless so the difference between W and W_s is entirely due to mechanical friction losses occurring in the mechanism. This difference depends not only on the effectiveness of the engine mechanism, but also on the force loading applied. This loading is strongly influenced by the choice of buffer pressure.

C. *Role of buffer pressure in energy transfers*

Figure 7 illustrates the effect of buffer pressure level on the nature of the work transfers that must be effected through the mechanism. Shown is an elliptical engine cycle with a sequence of increasing buffer pressures. A plus sign indicates a portion of the cycle where work is transferred from the piston to the flywheel/output shaft; note that these parts of the cycle are precisely where either the workspace pressure is above buffer pressure and the piston is effecting an expansion, or where the workspace pressure is below buffer level and a compression is taking place. A

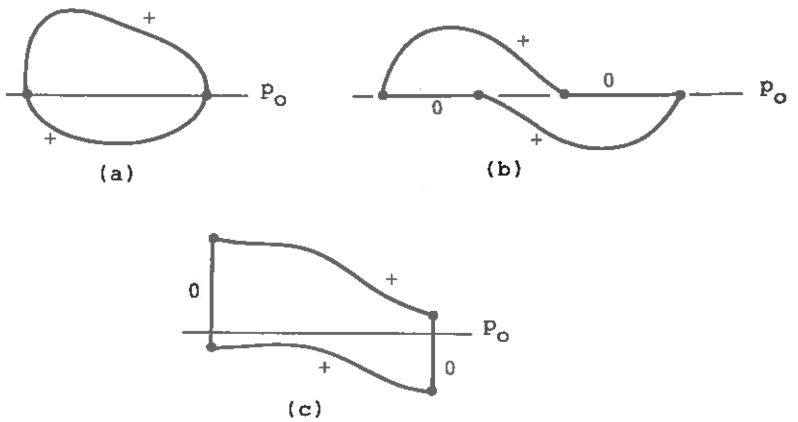

FIG. 8. Examples of regular cycles which require no flywheel for certain buffer pressure levels.

minus sign signifies the opposite situation, namely, a process in which work must be transferred from the flywheel to the piston. Note that the signs do not necessarily relate to the direction of piston or mechanism motion, but rather only differentiate the direction of positive work transfer between the piston and the flywheel/output shaft across the mechanism.

These exchanges and the corresponding mechanical losses will be quantified in the next section, but some important points should be intuitively clear at this juncture. One is that an engine buffered as in (a) will require more "flywheel effect" than that in (b); case (c) will require even less and (d) near minimal. Accordingly, larger mechanical friction losses will occur in case (a) than in (b) etc. Higher buffer pressures as in (e), (f) and (g) have effects similar to the lower pressures.

It should also be clear that no choice of buffer pressure level will entirely eliminate the need for a flywheel for this particular cycle. At any buffer pressure it will have some segment labelled with a minus sign and so some flywheel energy storage is essential for this cycle. This is typical but not always the case, as Fig. 8 shows. In each cycle in Fig. 8 there is a buffer pressure which results in positive work transfer only from the piston to the output shaft, that is, no "minus" segments occur.

In Fig. 8(a) and (b), the only "non-plus" points occur when the cycle pressure equals the buffer pressure; these points are indicated by a dot or by the label "0". Since there is no pressure difference across the piston in these locations, no force need be applied to or from the piston so no work transfer takes place.

In Fig. 8(c), vertical segments occur at the volume extremes. We adopt the convention that no work transfer occurs on such segments since no piston motion takes place there. Now in practice when such dwells occur, some parts of the engine may remain in motion (e.g. the displacers in Stirling engines) so there is some associated friction loss. However, this loss is generally small compared to other losses during the non-isometric portions of the cycle so it will be ignored here in the interest of obtaining general results. When vertical segments do occur, there may be a range of buffer pressures for which $W_{-} = 0$, as is the case in Fig. 8(c).

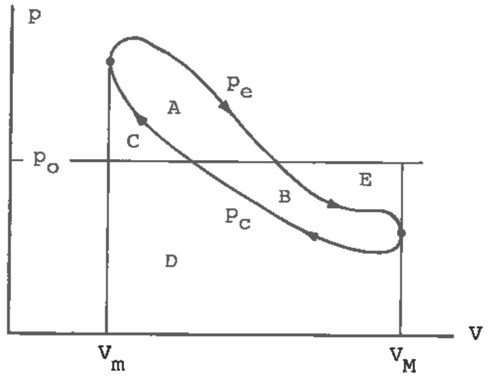

FIG. 9. The work components of a regular cycle.

D. *Work components*

The following definition identifies basic work components of a regular cycle with buffer pressure p_o:

$$A\text{-work} = \text{area within the cycle and above } p_o,$$

$$B\text{-work} = \text{area within the cycle and below } p_o,$$

$$C\text{-work} = \text{area below } p_c \text{ and above } p_o,$$

$$D\text{-work} = \text{area below } p_c \text{ and below } p_o,$$

$$E\text{-work} = \text{area above } p_e \text{ and below } p_o.$$

The regions corresponding to these quantities are illustrated in Fig. 9. Note that all are non-negative by definition. In this terminology, the works relevant to engine operation find easy expression:

$$\begin{aligned} W &= A+B, \\ W_e &= A+B+C+D, \\ W_c &= C+D, \\ W_+ &= A+B+C+E, \\ W_- &= C+E. \end{aligned} \tag{1}$$

IV. General Results

A. *The central theorem*

The positive piston work W_+ is the work input into the mechanism for transmittal to the flywheel/output shaft. In transit, some is lost to friction and the reduced amount W_o is actually delivered. A detailed knowledge of the effectiveness function e of the engine's mechanism is needed to determine W_o. Suppose we know, as is often the case in practice, constant bounds on the instantaneous effectiveness function. Suppose it is known that the effectiveness is at least h but at most k throughout the cycle:

$$0 < h \leqslant e \leqslant k \leqslant 1.$$

Then clearly,

$$hW_+ \leqslant W_o \leqslant kW_+. \tag{2}$$

The negative piston work W_- is work that must come out of the mechanism to sustain the cycle. It originates from work W_i input to the mechanism from the flywheel. Therefore, reasoning as above,

$$hW_i \leqslant W_- \leqslant kW_i. \tag{3}$$

It is not necessarily assumed here that the mechanism has the same effectiveness when working in both directions and for both input/output side orientations, but only that the h and k bounds cover operation in all four of these modes. In cases where there is a marked difference in the characteristics of the mechanism when working in different modes, each can be treated separately to improve accuracy of the result (at the expense of greater complication in its statement). Since $W_s = W_o - W_i$, inequalities (2) and (3) yield

$$hW_+ - \frac{W_-}{h} \leqslant W_s \leqslant kW_+ - \frac{W_-}{k}. \tag{4}$$

Using Eqs (1) in (4) yields the following bounds on the cyclic shaft work:

$$hW - [(1/h) - h](C+E) \leqslant W_s \leqslant kW - [(1/k) - k](C+E).$$

This result is summarized in the following theorem in terms of mechanical efficiency.

Theorem I. If the effectiveness of an engine mechanism is at least $h > 0$ and at most k throughout its cycle, then

$$h - \left(\frac{1}{h} - h\right)\frac{C+E}{W} \leqslant \eta_m \leqslant k - \left(\frac{1}{k} - k\right)\frac{C+E}{W}.$$

Since $C+E$ depends upon cycle shape and the choice of buffer pressure, this result shows how these two characteristics influence mechanical efficiency. In Fig. 10 is shown the cycle of Fig. 7 for various buffer pressures with the associated C- and E-work areas shaded. Note that $E = 0$ in (a), (b) and (c), while in (e), (f) and (g), $C = 0$. It should be noted that this theorem also holds in the case of non-constant buffer pressure with hysteresis loss (6).

As an example of the straightforward application of this theorem, suppose it is known that the effectiveness of the mechanism of a certain engine is between 0.8 and 0.9. Suppose also that a buffer pressure is chosen which makes the $C+E$ work equal to 0.25 of the indicated work W. Then Theorem I predicts that the mechanical efficiency cannot exceed 0.85 and may be as low as 0.69:

$$0.69 \leqslant \eta_m \leqslant 0.85.$$

Such applications can be useful for quick estimates in practical situations.

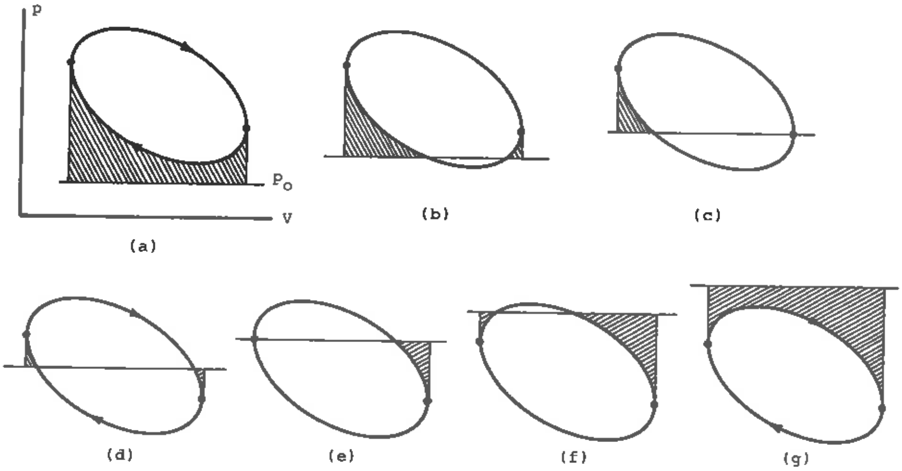

FIG. 10. The influence of buffer pressure level on the C- and E-work of a regular cycle. The shaded area in each case represents the total $C+E$ work.

B. *Constant mechanism effectiveness*

From Theorem I the following result is immediate.

Theorem II. If the mechanism effectiveness of a kinematic engine is a constant e, then

$$\eta_m = e - \left(\frac{1}{e} - e\right)\frac{C+E}{W}.$$

The conclusion of this theorem stated in terms of shaft work is:

$$W_s = eW - [(1/e) - e](C+E).$$

Typical engine mechanisms have effectiveness functions that are non-constant. Yet, in many cases, assuming a constant effectiveness is a sufficiently good approximation for the results desired. This is certainly the case when the bounds in Theorem I are fairly close together. For example, taking $e = 0.85$ in the above hypothetical engine, Theorem II yields a mechanical efficiency of 0.77, which is the average of the limits given by Theorem I. Moreover, the assumption of constant effectiveness is appropriate for the general nature of this study where the focus is on obtaining insight into the relations between the most basic specifications of an engine and its mechanical efficiency. More detailed investigations must take the specific mechanism as well as the p–V cycle as the starting point. For the remainder of the paper, the mechanism effectiveness will be assumed constant. Note that Theorem II shows that $\eta_m \leqslant e$ with equality holding if and only if $e = 1$, which is without interest, or the buffer pressure is such that $C = 0 = E$. Cycles with the property that for some buffer pressure its C and E works are both zero will be called *efficacious*. The cycles shown in Fig. 8 are examples. Efficacious cycles, with the right buffer pressure, have a mechanical efficiency equal to the maximum possible, namely the effectiveness of their mechanism.

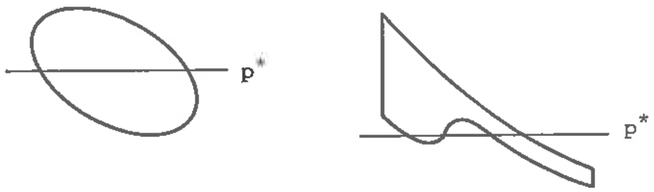

FIG. 11. Examples of regular cycles and their optimum buffer pressure.

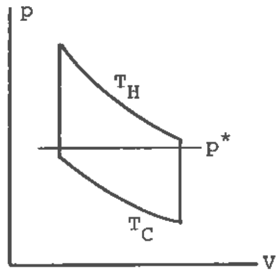

FIG. 12. An ideal Stirling cycle with $r\tau \leqslant 1$.

C. *Optimum buffer pressure*

It follows from the definition of a regular cycle that its C-work and E-work are continuous functions of buffer pressure. Moreover, $C+E$-work is bounded below by 0, and increases without bound as buffer pressure increases above and beyond the entire cycle. Therefore, $C+E$ has an absolute minimum at some buffer pressure level. By Theorem II, this pressure is *optimum* for the engine in the sense that it gives the maximum mechanical efficiency possible, or equivalently, the maximum cyclic shaft work.

Theorem III. Every kinematic engine has an optimum buffer pressure.

Note that the optimum buffer pressure is a characteristic of the engine cycle alone and does not depend upon the mechanism's effectiveness. Figure 11 shows two regular cycles and their optimum buffer pressures p^*.

V. Optimally Buffered Stirling Engines

The above concepts and results are now applied to Stirling engines. An *ideal Stirling cycle* is defined as a cycle comprised of two isotherms and two isometrics. Let τ represent the *temperature ratio* of the isothermals of the cycle and let r represent the *volume compression ratio*:

$$\tau = T_C/T_H \quad \text{and} \quad r = V_M/V_m.$$

By an *ideal Stirling engine* we mean a kinematic heat engine with an ideal working gas which undergoes an ideal Stirling cycle. In this section, an explicit expression is given for the mechanical efficiency of ideal Stirlings buffered at the optimum pressure.

Some Stirling cycles are efficacious, that is, $(C+E)_{\min} = 0$, as Fig. 12 shows. These are Stirling cycles where the peak pressure on the lower isotherm T_C is not

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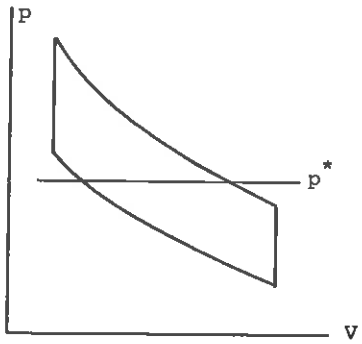

FIG. 13. An ideal Stirling cycle with $r\tau > 1$.

larger than the minimum pressure on the upper isotherm T_H. Any pressure between these extremes is an optimum buffer pressure in this case. It is easy to show that a Stirling is efficacious if and only if $\tau r \leqslant 1$.

The case when $\tau r > 1$ is shown in Fig. 13. The optimum buffer pressure p^* in this case can be readily found by equating the base lengths of the C and E regions, or by elementary differential calculus. We find

$$p^* = \frac{mRT_H}{V_m}\frac{1+\tau}{1+r}$$

from which $(C+E)_{\min}$ can be calculated. In this expression, m is the mass of gas in the workspace and R is the ideal gas constant. The results of both cases are summarized in the following theorem.

Theorem IV. If e is the mechanism effectiveness of an optimally buffered ideal Stirling engine with temperature ratio τ and volume compression ratio r, then its mechanical efficiency is given by

$$\eta_{ms} = e-\left(\frac{1}{e}-e\right)S(\tau, r)$$

where

$$S(\tau, r) = \begin{cases} 0 & \text{if } \tau r \leqslant 1 \\ \dfrac{\tau \ln \tau-(1+\tau)\,[\ln(1+\tau)-\ln(1+r)]-\ln r}{(1-\tau)\ln r} & \text{if } \tau r > 1 \end{cases}$$

Many practical Stirling engines operate with a buffer pressure equal to mean cycle pressure. Because some degree of leakage between the workspace and the buffer space is unavoidable, this is the "automatically" established buffer pressure of Stirling engines without special check valves or pumps. It is interesting to note how close this is to the optimum buffer pressure p^* given above.

For a regular cycle with *swept volume* $\Delta V = V_M - V_m$ we have made (7) the following definitions:

$$p_{mu} = W_e/\Delta V = \text{mean upper pressure},$$

$$p_{ml} = W_c/\Delta V = \text{mean lower pressure},$$

$$p_m = (p_{mu}+p_{ml})/2 = \text{mean cycle pressure}.$$

Applying these concepts to an ideal Stirling cycle, we get

$$p_m = \frac{mRT_H(\tau+1)\ln r}{2V_m(r-1)}.$$

Therefore,

$$\frac{p_m}{p^*} = \frac{(1+r)\ln r}{2(r-1)}.$$

Analysis shows $p_m > p^*$ since $r > 1$, and calculations summarized in the table below show p_m is very close to p^* for compression ratios typically found in practical Stirling engines:

r	p_m/p^*
1.1	1.00076
1.2	1.00277
1.3	1.00573
1.5	1.01366
2.0	1.03972
3.0	1.09861

Theorem IV explicitly describes the influence of the temperature ratio, compression ratio, and mechanism effectiveness on the mechanical efficiency of ideal Stirling engines. It is obviously relevant to Stirling engine theory and the new insights it affords explain many observations and intuitions based on the performance of actual Stirling engines (**8**). But the theorem also turns out to be of general importance for the entire field of kinematic heat engines, as shown in the next section.

VI. Universal Efficiency Theorems

In comparing engine performance, it is customary to compare engines having the same minimum and maximum temperatures, the same overall size, and the same general mechanical characteristics. The following theorems show that in all such comparisons, the Stirling engine has the highest efficiency potential.

A. *Limits to mechanical efficiency*

Theorem V. Of all kinematic heat engines having the same effective temperature extremes, the same volume extremes, the same mechanism effectiveness, and the same buffer pressure, the ideal Stirling has the maximum mechanical efficiency, and, if non-efficacious, is the only cycle attaining this maximum.

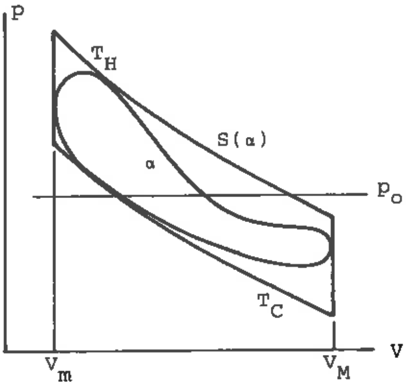

FIG. 14. A regular cycle α inscribed in its Stirling covering cycle $S(\alpha)$.

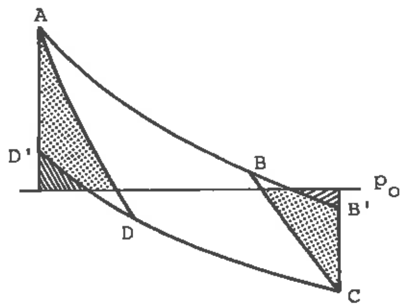

FIG. 15. A Carnot cycle and its Stirling covering cycle.

The proof of this theorem uses Theorem II and the idea of a Stirling covering cycle. Given any regular cycle α in the p–V plane, there is a unique ideal Stirling cycle $S(\alpha)$ inscribing α; this is illustrated in Fig. 14. We will refer to $S(\alpha)$ as the *Stirling covering cycle* of α. Both cycles have the same volume extremes, and with the same mass of the same kind of ideal gas in $S(\alpha)$ as in α, both cycles have the same effective temperature extremes. With a common buffer pressure, it is easy to see that the E-work of α is at least equal to the E-work of $S(\alpha)$. The same is true of the C-work. Hence the $C+E$-work of $S(\alpha)$ is less than or equal to that of α. Since $S(\alpha)$ circumscribes α, its indicated work W is greater than or equal to that of α. Therefore, $(C+E)/W$ for $S(\alpha)$ is less than or equal to that for α. With the same mechanism effectiveness for both engines, it follows by Theorem II that the mechanical efficiency of $S(\alpha)$ is greater than or equal to that of α. Moreover, equality holds if and only if α is efficacious or $\alpha = S(\alpha)$, for if α differs from $S(\alpha)$, then the indicated work of α is strictly less than for $S(\alpha)$, so its mechanical efficiency is less by Theorem II. This proves Theorem V.

Figure 15 illustrates the idea of the above proof by showing a Carnot cycle $ABCD$ inscribed in its Stirling cover $AB'CD'$. The line-shaded areas are the C- and E-work of the Stirling cover. The $C+E$-work of the Carnot cycle is represented by both the line-shaded plus the dot-shaded areas.

Theorem V is all the more true if the Stirling covering cycle is optimally buffered. In this case, by Theorem IV, only the temperature and volume ratios, τ and r, together with e, determine the mechanical efficiency. We therefore immediately get the following, more general, theorem.

Theorem VI. $\eta_{ms} = e-[(1/e)-e]S(\tau,r)$ is an upper bound to the mechanical efficiency of all kinematic heat engines with the same e, τ, and r values. Moreover, an engine with the same e, τ, and r values has this mechanical efficiency only if it is an optimally buffered ideal Stirling engine whenever $\tau r > 1$.

This theorem is important because it places a mathematically explicit limit on the mechanical efficiency of arbitrary kinematic heat engines in terms of only three conceptually simple dimensionless characteristics of the engine. η_{ms} is to mechanical efficiency what $1-\tau$ is to thermal efficiency. It is an in-principle limit that no kinematic engine can exceed and that only ideal Stirling engines can equal. In the next section, this result is seen to extend to cover combined thermal and mechanical efficiency.

B. *Limits to brake thermal efficiency*

The *brake thermal efficiency* of a kinematic heat engine is defined as the ratio of its cyclic shaft work to its cyclic heat absorption; in symbols:

$$\eta_b = W_s/Q_i.$$

Equivalently, brake thermal efficiency is the product of mechanical efficiency, η_m, and thermal efficiency, η_t.

It is well known that the upper limit to the thermal efficiency of heat engines is that of the Carnot, namely, $1-\tau$. It is less well known, however, that the Carnot cycle is not unique in having this property. There is an entire class of cycles that potentially have the Carnot thermal efficiency. These are the so-called Reitlinger cycles, which are defined by two isothermals and two polytropics of the same kind (9). In such cycles, the quantity of heat absorbed on one polytropic is exactly equal to that rejected on the other, so that regeneration is possible in principle. With perfect regeneration, the thermal efficiency of a Reitlinger cycle equals that of the Carnot, which is actually a special Reitlinger cycle in which the polytropics are adiabatics.

The ideal Stirling is also a Reitlinger cycle, it being a limiting case in which the polytropic segments are isometrics. Therefore, with perfect regeneration the thermal efficiency of the Stirling is maximal. So also is its mechanical efficiency by our Theorem VI. This gives the following theorem.

Theorem VII. If η_b is the brake thermal efficiency of a kinematic heat engine with constant mechanism effectiveness e, effective temperature ratio τ, and volume compression ratio r, then

$$\eta_b \leqslant (1-\tau)\left[e-\left(\frac{1}{e}-e\right)S(\tau,r)\right] = (1-\tau)\eta_{ms}.$$

If $\tau r > 1$, then equality holds only if the engine is an optimally buffered ideal regenerative Stirling engine.

This theorem shows that in total brake thermal efficiency, the ideal Stirling surpasses the Carnot and all other possible engines.

VII. Conclusion

The concepts and results presented in this paper are given in their most fundamental and idealized form in the interest of universality and accessibility. Specialized results can be found in references (**6**, **8**, **10**), where practical applications are given further attention.

Acknowledgement

The initial stages of the work reported here were carried out while the author was on a sabbatical at Argonne National Laboratory. He is deeply grateful for Argonne's support during this period and for continuing support of this work under National Science Foundation Grant No. 8705576.

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MECHANICAL EFFICIENCY OF KINEMATIC HEAT ENGINES *

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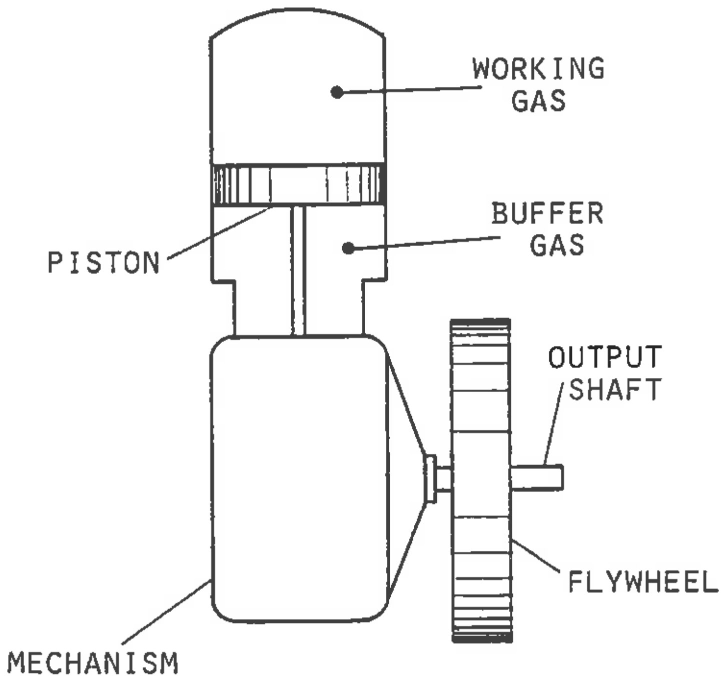
Basic elements of a kinematic heat engine



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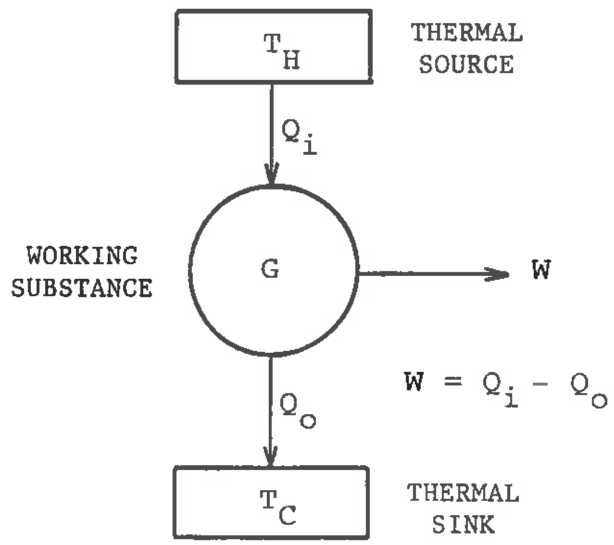
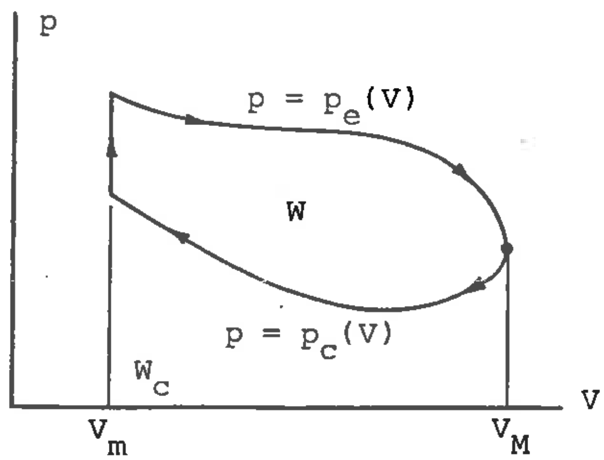
General Heat Engine Diagram



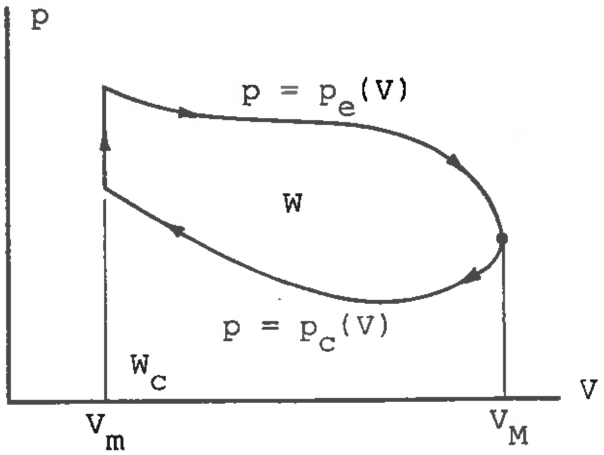
P-V cycle of the working gas



Regular cycle:

$p_e(V) \geq p_c(V)$

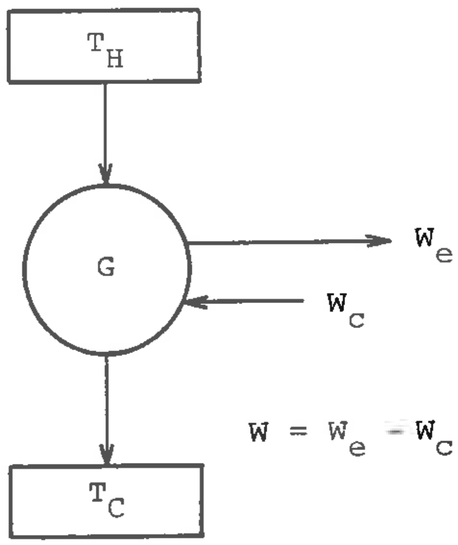
$V_m \leq V \leq V_M$



Work exchanges with engine fluid

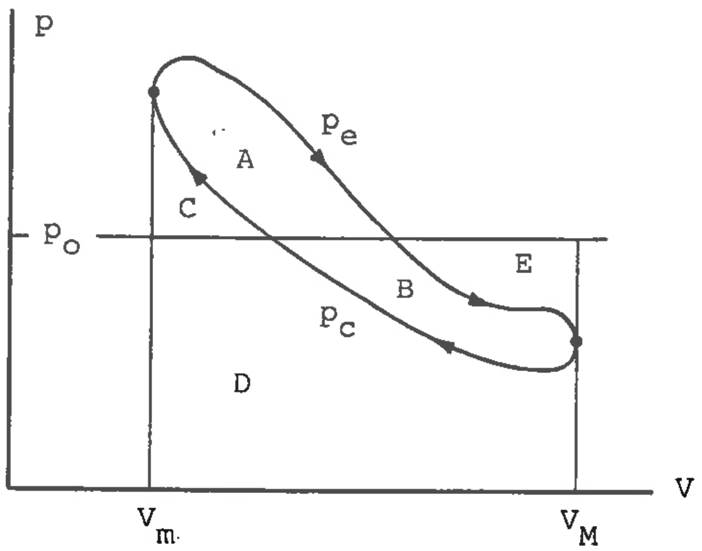
$$W_e = \int_{V_m}^{V_M} p_e(V)\,dV$$

$$W_c = \int_{V_m}^{V_M} p_c(V)\,dV$$

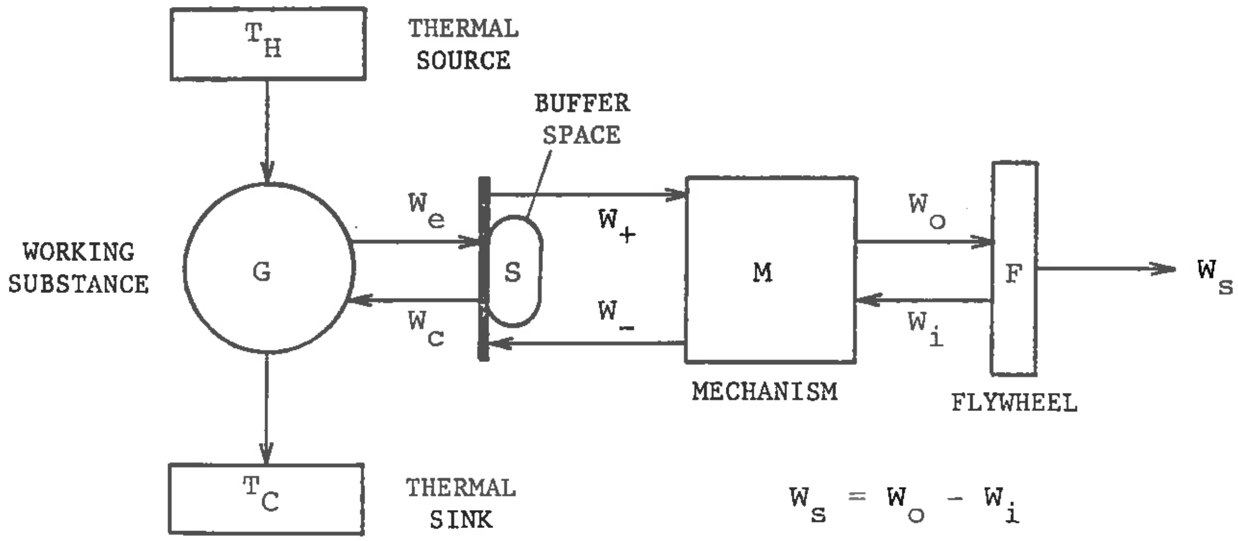


Indicated cyclic work $= W = W_e - W_c$

Work components of a regular cycle



Indicated work	$W = A + B$
Expansion work	$W_e = A + B + C + D$
Compression work	$W_c = C + D$
Positive piston work	$W_+ = A + B + C + E$
Negative piston work	$W_- = C + E$



Mechanical Efficiency

$$\eta_m = \frac{W_s}{W} = \frac{\text{cyclic work at output shaft}}{\text{indicated cyclic work}}$$

Mechanism Effectiveness

$$e = \frac{\text{actual output force}}{\text{ideal output force}}$$

Mechanical Efficiency Theorem

If the effectiveness of an engine mechanism is at least h and at most k throughout the cycle, then

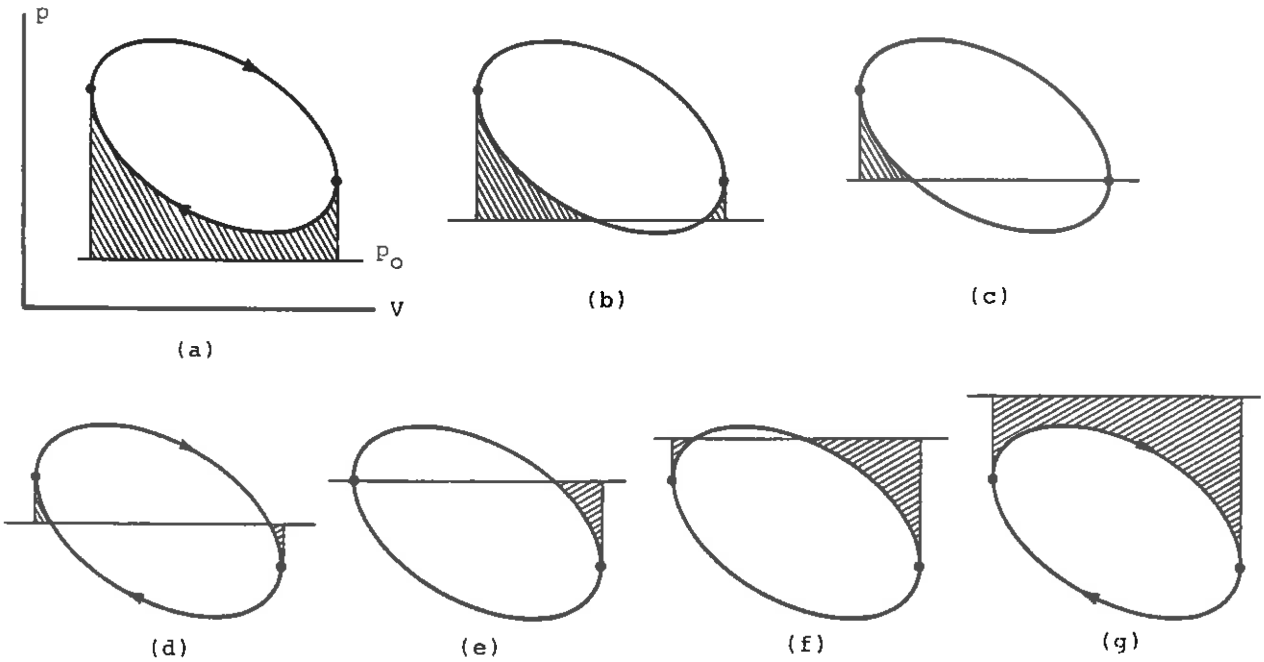
$$h-\left(\frac{1}{h}-h\right)\frac{C+E}{W} \leq \eta_m \leq k-\left(\frac{1}{k}-k\right)\frac{C+E}{W}$$

Corollary

If the mechanism effectiveness of an engine is the constant e, then

$$\eta_m = e-\left(\frac{1}{e}-e\right)\frac{C+E}{W}$$

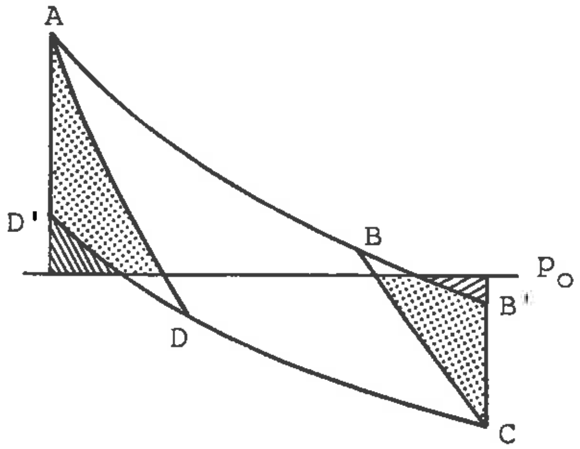
Optimum Buffer Pressure



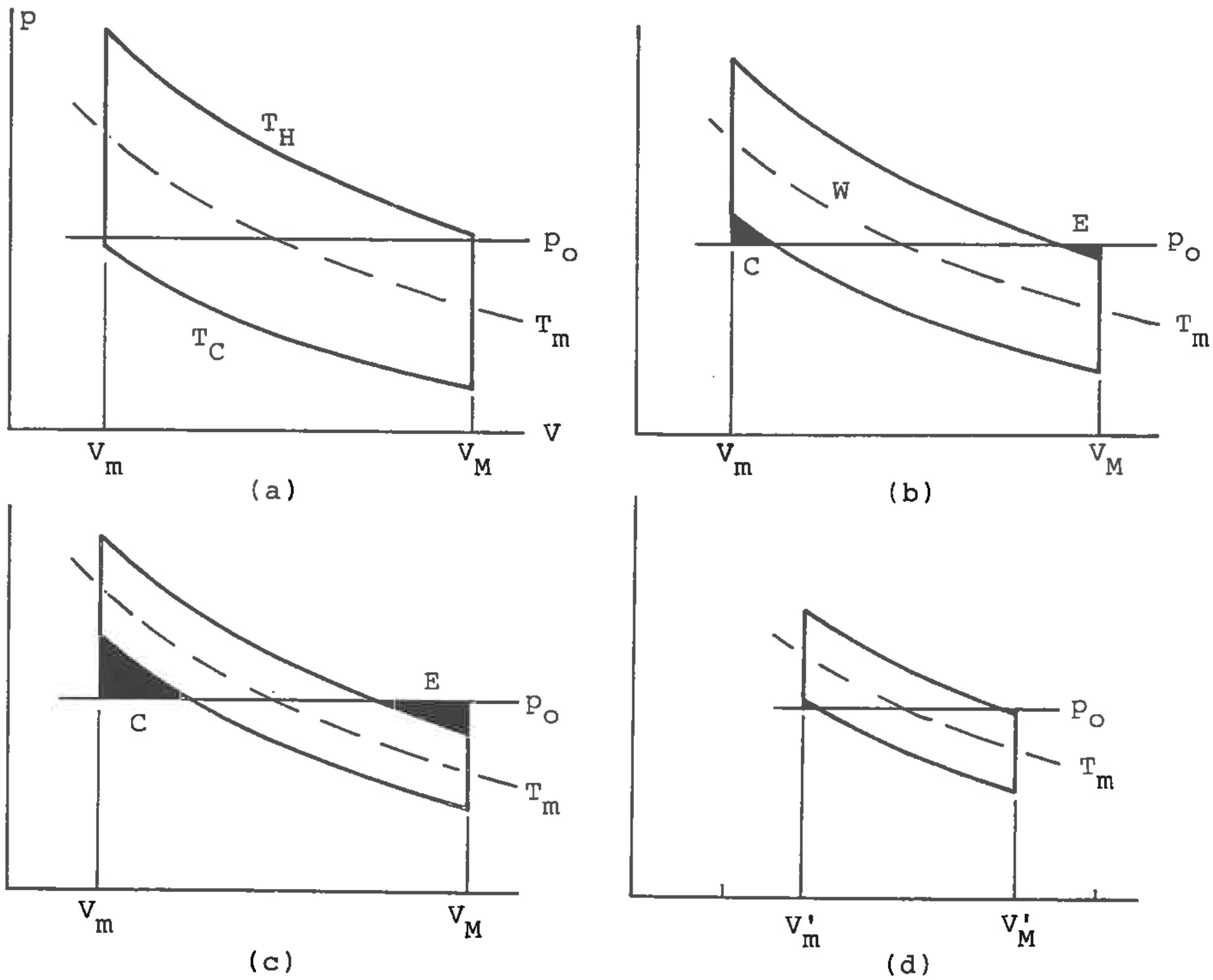
Theorem

Every kinematic engine has an optimum buffer pressure.

Comparison of the C and E -works of a Carnot cycle and a Stirling cycle.

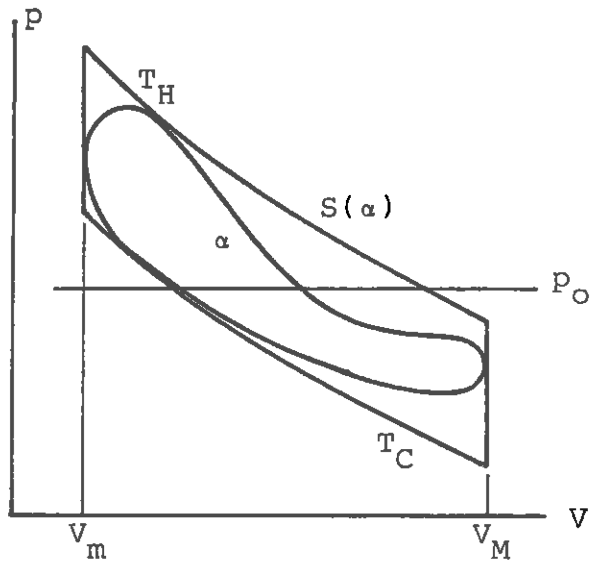


Behavior of an ideal Stirling engine as operating temperatures degrade.



Stirling Covering Cycle

A regular cycle inscribed in its Stirling covering cycle.



Theorem

Of all kinematic heat engines having the same effective temperature extremes, the same volume extremes, the same mechanism effectiveness, and the same buffer pressure, the ideal Stirling has the maximum mechanical efficiency, and is the only cycle attaining this maximum.

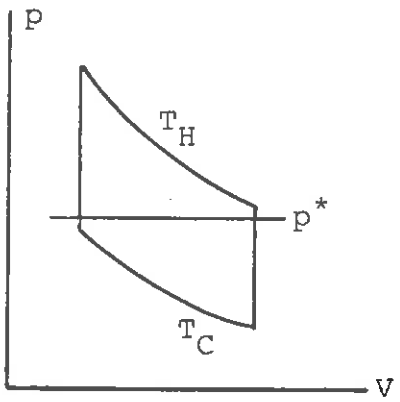
Temperature Ratio

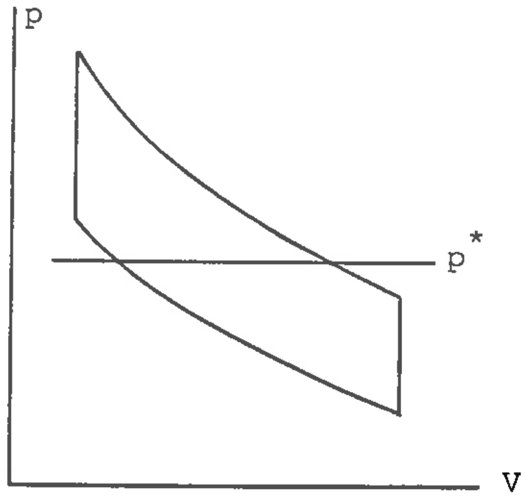
$$\tau = \frac{T_C}{T_H}$$

Compression Ratio

$$r = \frac{V_M}{V_m}$$

Optimally Buffered Stirling Engines





$\tau r \leq 1$

$$C + E = 0$$

$\tau r > 1$

$$p^* = \frac{mR\,(T_H + T_C)}{V_m + V_M}$$

Efficiency of Optimally Buffered Ideal Stirling Engine

Theorem

If e is the mechanism effectiveness of an optimally buffered ideal Stirling engine with temperature ratio τ and volume compression ratio r, then its mechanical efficiency is given by

$$\eta_{ms} = e - \left(\frac{1}{e} - e\right) S(\tau,r)$$

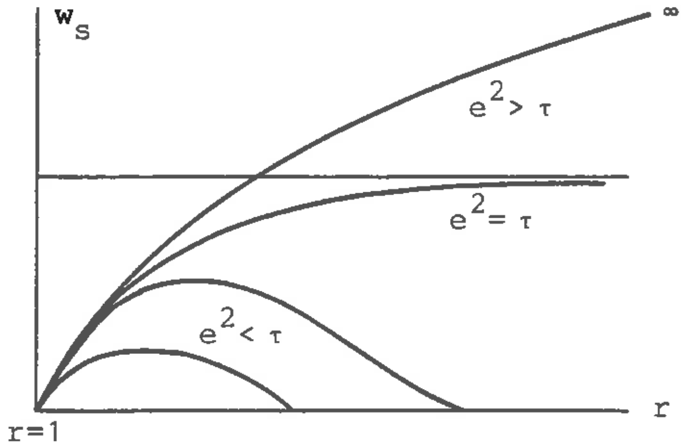
where

$$S(\tau,r) = \begin{cases} 0 \quad \text{if } \tau r \le 1 \\ \dfrac{\tau \ln \tau - (1+\tau)[\ln(1+\tau) - \ln(1+r)] - \ln r}{(1-\tau)\ln r} \\ \text{if } \tau r > 1. \end{cases}$$

These two theorems show that the efficiency function $\eta_{ms}(e,\tau,r)$ given above is an upper bound to the mechanical efficiency of <u>all</u> kinematic heat engines with the same e, τ and r values.

APPLICATIONS

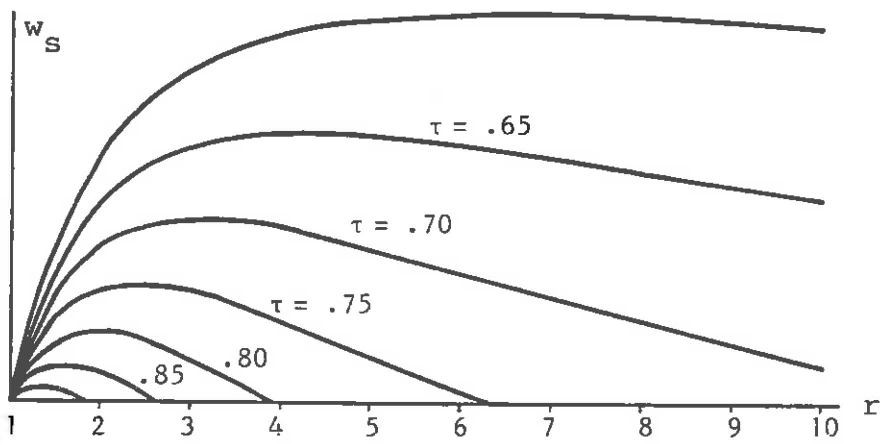
Behavior of specific shaft output w_s of an ideal optimally buffered Stirling engine as a function of compression ratio r.



General Principle I

Given e and τ with $e^2 < \tau$, there is a compression ratio value above which no kinematic heat engine with these e and τ values will operate.

Graph of specific shaft output of an ideal optimally buffered Stirling engine with mechanism effectiveness e = .70.



General Principle 2

For the same mechanism effectiveness, engines operating across small temperature differences require lower compression ratios then do engines operating between larger temperature differences.